MATH 6021 Lecture 5 10/5/2020

Monotonicity Formula:

Let $\Sigma^k \subset \mathbb{R}^n$ be an (immersed) min. submanifold.

Fix $x_0 \in \mathbb{R}^n$ (not nec. in Σ), consider $B_r := B_r(x_0) =$ open ball of radius $r > 0$ centered at x_0

Then, $\forall\ 0 < s < t < d(x_0, \partial\Sigma)$,

$$\frac{|\Sigma \cap B_t|}{t^k} - \frac{|\Sigma \cap B_s|}{s^k} = \int_{\Sigma \cap (B_t \setminus B_s)} \frac{|(x - x_0)^N|^2}{|x - x_0|^{k+2}} \quad (\geq 0)$$

Remark: The formula holds for "singular" min. submfd (currents or varifolds) and slightly "perturbed" in the Riemannian setting.

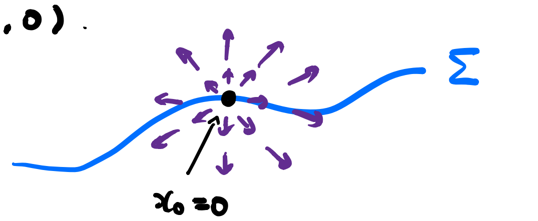
Proof: (L. Simon "Lectures on GMT"; C.M. Ch. 3)

W.L.O.G., take $x_0 = 0$.

Recall: $\delta\Sigma(X) = \int_\Sigma \operatorname{div}_\Sigma X = 0 \quad \forall$ cpt. supp. vector field X in $\mathbb{R}^n$.

Idea: Choose X to be certain cutoff of radial vector field

Take $X(x) := \overbrace{\gamma(r)}^{\text{cutoff}}\ \overbrace{x}^{\text{radial vec. field}}$ where $r := |x| = \operatorname{dist}^{\mathbb{R}^n}(x, 0)$.



Compute the divergence,

$$\operatorname{div}_\Sigma X = \sum_{i=1}^k \nabla^{\mathbb{R}^n}_{e_i} X \cdot e_i$$

$$= k\,\gamma(r) + \gamma'(r) \sum_{i=1}^k \overbrace{(\nabla^{\mathbb{R}^n}_{e_i} r)}^{\frac{x \cdot e_i}{r}} (x \cdot e_i)$$

$$= k\,\gamma(r) + r\,\gamma'(r)\, \underbrace{|\nabla^T r|^2}_{= 1 - |\nabla^N r|^2}$$

$$\nabla^{\mathbb{R}^n} r = \frac{x}{r}$$

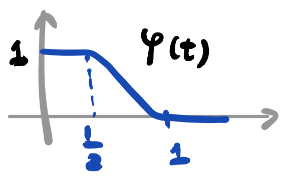
Integrate over Σ, by first variation formula,

$$0 = \int_\Sigma \operatorname{div}_\Sigma X = k\int_\Sigma \gamma(r) + \int_\Sigma r\gamma'(r) - \int_\Sigma r\gamma'(r)\,|\nabla^N r|^2$$

i.e. $$k\int_\Sigma \gamma(r) + \int_\Sigma r\gamma'(r) = \int_\Sigma r\gamma'(r)\,|\nabla^N r|^2 \qquad \cdots\cdots (1)$$

Choose $\gamma(r) = \varphi\left(\frac{r}{\rho}\right)$ where $\rho > 0$ is some "parameter"

where φ = cutoff fcn



<u>Note:</u> $\gamma \equiv 1$ in $B_{\rho/2}$ and $\gamma \equiv 0$ outside B_ρ

"<u>Note</u>": $r\,\frac{d}{dr}\gamma(r) = -\rho\,\frac{d}{d\rho}\left(\varphi\left(\frac{r}{\rho}\right)\right)$ by Chain Rule

(1) becomes

$$k\underbrace{\int_\Sigma \varphi\left(\frac{r}{\rho}\right)}_{I(\rho)} - \rho\,\frac{d}{d\rho}\left(\underbrace{\int_\Sigma \varphi\left(\frac{r}{\rho}\right)}_{I(\rho)}\right) = -\rho\,\frac{d}{d\rho}\int_\Sigma \varphi\left(\frac{r}{\rho}\right)|\nabla^\perp r|^2$$

Define $I(\rho) := \int_\Sigma \varphi\left(\frac{r}{\rho}\right)$. We have

$$\frac{d}{d\rho}\left(\rho^{-k} I(\rho)\right) = \rho^{-k} I'(\rho) - k\,\rho^{-k-1} I(\rho)$$

$$= -\rho^{-k-1}\left[k\,I(\rho) - \rho\,I'(\rho)\right]$$

"Let $\varphi(t) \to \chi_{[0,1]}$": $$= \rho^{-k}\,\frac{d}{d\rho}\left(\int_\Sigma \varphi\left(\frac{r}{\rho}\right)|\nabla^\perp r|^2\right)$$

Then. $I(\rho) = |\Sigma \cap B_\rho|$.

$$\frac{d}{d\rho}\left(\rho^{-k}|\Sigma\cap B_\rho|\right) = \rho^{-k}\frac{d}{d\rho}\left(\int_{\Sigma\cap B_\rho}|\nabla^\perp r|^2\right) \overset{\text{coarea for.}}{=} \frac{d}{d\rho}\left(\int_{\Sigma\cap B_\rho}\frac{|\nabla^\perp r|^2}{r^k}\right)$$

$$|\nabla^\perp r|^2 = \frac{|x^\perp|^2}{|x|^2}$$

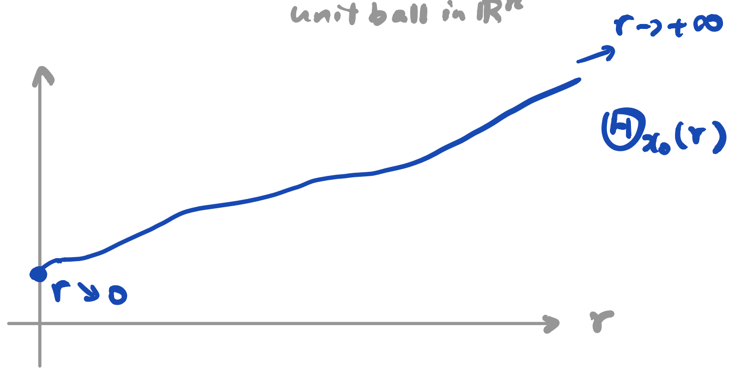
Some consequences of Monotonicity Formula

(i) The "volume ratio"

$$\Theta_{x_0}(r) := \frac{|\Sigma \cap B_r(x_0)|}{\sigma_k r^k} \quad \text{is non-decreasing in } r.$$

σ_k: Vol. of k-dim. unit ball in $\mathbb{R}^k$

E.g.)



(ii) The limits on both sides exist (even maybe $+\infty$)

$$\Theta_{x_0} := \lim_{r \searrow 0} \Theta_{x_0}(r) < +\infty \quad \leftarrow \text{density of } \Sigma \text{ at } x_0$$

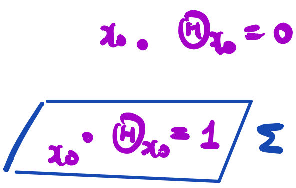
$$\Theta_{\Sigma}^{\infty} := \lim_{r \nearrow +\infty} \Theta_{x_0}(r) \leq +\infty \quad \leftarrow \text{density of } \Sigma \text{ at } \infty$$

[Note: $\Theta_{\Sigma}^{\infty} < +\infty \iff \Sigma$ has Euclidean volume growth]

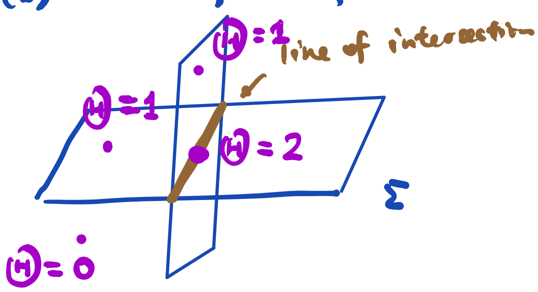
(iii) The density function $x_0 \longmapsto \Theta_{x_0}$ is upper semi-continuous.

Examples / Pictures:

(a) Plane



(b) Union of two planes



Remarks:

- $\Theta_{x_0} = 1$ at any "regular" embedded pt on Σ
- Θ plays a main role in the regularity theory

Existence Theory for Minimal Surfaces

Q: How to construct minimal surfaces in $\mathbb{R}^n$ or (M^n, g)?

First, look at $\mathbb{R}^n$, even $n=3$ ......

Recall: max. principle $\Rightarrow$ $\nexists$ closed (cpt w/o bdy) min. submfd in $\mathbb{R}^n$

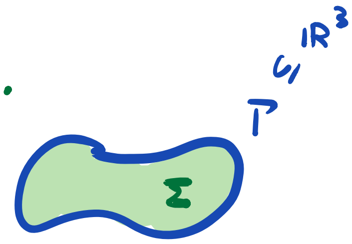
So, we are interested in:

Global (1) complete, non-cpt min. surfaces (E.g. plane, catenoid, helicoid)

Local (2) min. surfaces with boundary (E.g. disk)

Plateau's Problem ("Dirichlet BVP for min. surfaces")

Given a simple closed (Jordan) curve $\Gamma \subseteq \mathbb{R}^3$.



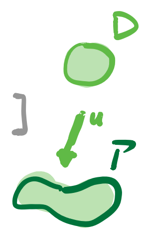
- $\exists$ min. surface $\Sigma \subseteq \mathbb{R}^3$ with $\partial\Sigma = \Gamma$?
- Is there an "area-minizing" Σ?

Subtlety: Depends very much on what "surfaces" are and how to measure their "area"?

Various approaches to Plateau's Problem

(1) PDE approach [Γ, Σ graphs $\rightsquigarrow$ Dirichlet BVP for (MSE)]

(2) "Parametrized" approach [Mapping problem: $u: D \to \mathbb{R}^3$, energy]

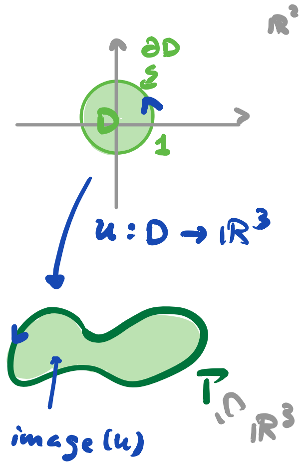


(3) GMT approach [weak surfaces eg. "currents"/"varifolds"]

(4) "Set-theoretic" approach [min. among "sets", Reifenberg '60s]

(5) "Capillary model" for min. surfaces [F. Maggi et al. ~ 2019-20]

Douglas-Rado Theorem: (Ref: C.M. Ch.4)



For any given piecewise C^1 Jordan curve $\Gamma \subseteq \mathbb{R}^3$,

$\exists$ a map $u : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ s.t.

(1) $u : \partial D \longrightarrow \Gamma$ is monotone & onto.

(2) $u \in C^0(\overline{D}) \cap W^{1,2}(D)$ and $u \in C^\infty(D)$

*(3) "image (u)" minimizes "area" among all parametrized disk w/ boundary Γ.

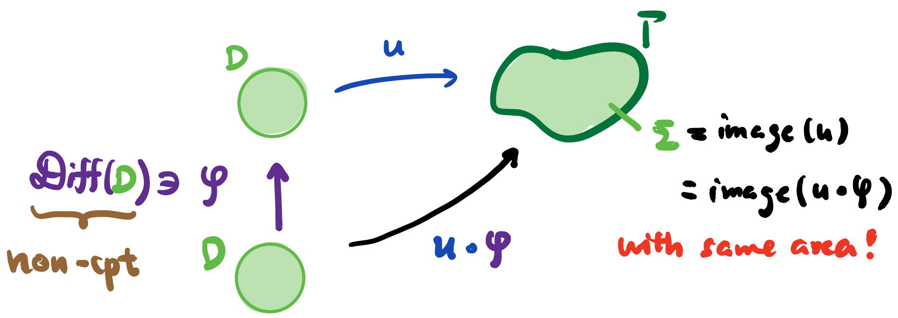
Remark: C. Morrey works in Riemannian manifold.

Q: Why is it difficult to prove a theorem like this?

Direct Method: Take a "minimizing seq." & pass to a sub-seq. limit

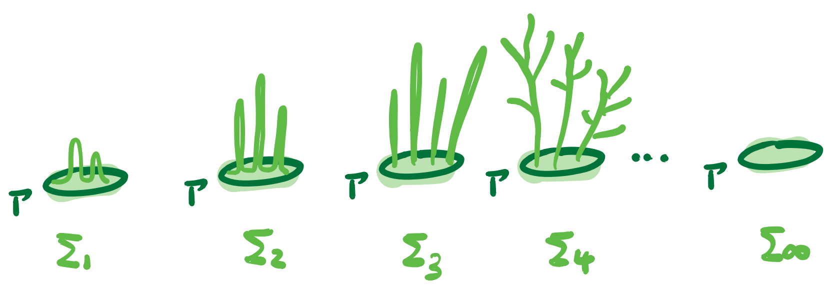
Difficulty 1

(Area is a "geometric" notion.)



NO compactness!

Difficulty 2



area (Σ_i) $\longrightarrow$ area (Σ_∞)

BUT $\Sigma_i \not\longrightarrow \Sigma_\infty$ (in C^0-sense)

Key Idea: Work with "energy" instead of "area".

Denote: $X_\Gamma := \{ u : D \to \mathbb{R}^3 \text{ satisfy (1), (2) in Douglas-Rado Thm} \}$

For each $u \in X_\Gamma = \{ \text{parametrized disk w/. bdy } \Gamma \}$, define:

Observation

$$\text{Area}(u) := \int_D \sqrt{|u_x|^2 |u_y|^2 - \langle u_x, u_y \rangle^2}\, dx\, dy$$ ← diffeomorphism invariant

$$\text{Energy}(u) := \frac{1}{2} \int_D |\nabla u|^2\, dx\, dy$$ ← conformally invariant

Let $A_\Gamma := \inf_{u \in X_\Gamma} \text{Area}(u)$ & $E_\Gamma := \inf_{u \in X_\Gamma} \text{Energy}(u)$.

Lemma: $A_\Gamma = E_\Gamma$.

"Proof": We have the pointwise inequality: (#)

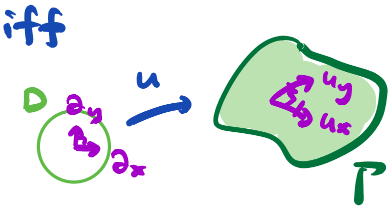
$$\left[\sqrt{|u_x|^2 |u_y|^2 - \langle u_x, u_y \rangle^2} \le |u_x||u_y| \le \frac{1}{2}\left(|u_x|^2 + |u_y|^2\right) = \frac{1}{2} |\nabla u|^2 \right]$$

Integrate over D, we get $\text{Area}(u) \le \text{Energy}(u)$ $\forall u \in X_\Gamma$.

This implies $A_\Gamma \le E_\Gamma$.

For $A_\Gamma \ge E_\Gamma$, we observe "=" holds in (#) iff

$$\langle u_x, u_y \rangle = 0 \quad \& \quad |u_x| = |u_y|$$



i.e. u is "conformal".

By the existence of (global) isothermal coordinates on $(D, u^* g_{\mathbb{R}^3})$

$\Rightarrow A_\Gamma \ge E_\Gamma$.

$\square$